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DYNAMIC EFFECT OF HIGH SPEED RAILWAY TRAFFIC LOADS ON THE BALLAST TRACK SETTLEMENT

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Abstract. *The traditional ballast track structures are still being used in high speed railways lines with success, however technical problems or performance features have led to non-ballast track solution in some cases. A considerable maintenance work is needed for ballasted tracks due to the track deterioration. Therefore it is very important to understand the mechanism of track deterioration and to predict the track settlement or track irregularity growth rate in order to reduce track maintenance costs and enable new track structures to be designed. The objective of this work is to develop the most adequate and efficient models for calculation of dynamic traffic load effects on railways track infrastructure, and then evaluate the dynamic effect on the ballast track settlement, using a ballast track settlement prediction model, which consists of the vehicle/track dynamic model previously selected and a track settlement law. The calculations are based on dynamic finite element models with direct time integration, contact between wheel and rail and interaction with railway cars. A initial irregularity profile is used in the prediction model. The track settlement law is considered to be a function of number of loading cycles and the magnitude of the loading, which represents the long-term behavior of ballast settlement. The results obtained include the track irregularity growth and the contact force in the final interaction of numerical simulation.*

1 INTRODUCTION

The advent and success of high speed railways and the increasing demand for sustainable development is enabling a comeback of railway transport, which is increasing the share in passenger traffic and perhaps also for freight traffic. This is a clear trend in Europe and Asia. An implication of this development is the need of new standards and regulations, which among other objectives must provide criteria for safety and functionality of new or existing railway infrastructure [6], [7], [11].

The requirements for the infrastructure, in the case of traditional ballasted track, represent a considerable amount, in the order of 60 k€ per km and year. It is of course a priority to optimize these costs and achieve a reduction in maintenance costs due to track quality deterioration and permanent settlements in ballasted track, which increase with velocity, traffic loads, number of load cycles and other factors. An essential ingredient for this is to be able to predict accurately these permanent settlements and track quality.

The deterioration of the track can be produced in different components of the track, generally these components may be categorized into two groups: the superstructure and the substructure. The settlement of the substructure is the main cause that produces the deterioration. The substructure of ballast track usually consists the ballast layer, a possible intermediate layer of sub-ballast and sub-grade. The ballast layer supports the track structure (rail and sleepers) and has several important functions:

- It limits sleeper movement by resisting vertical, transverse and longitudinal forces from the trains.
- It distributes the load from the sleepers to protect the sub-grade from high stresses, thereby limiting permanent settlement of track.
- It provides necessary resilience to absorb shock from dynamic loading.

However, it has not been yet reached a general consensus about adequate requirements for the ballast material index characteristics, such as particle size, particle shape, abrasion resistance and composition. Instead, the choice of a ballast material is commonly influenced by economic considerations and availability, where an extended variety of materials is used, such as a crushed granite, limestone, slag, and gravel. Figure 1 shows the typical profile of relative contribution of the substructure's components to the track settlement. The ballast layer contributes the most to track settlement, compared to subballast and sub-grade.

Several models exist for predicting the evolution of the permanent settlement of the track as a function of time or traffic loads. Most of these involve empirical expressions that depend on such factors as train speed, traffic loading, number of loading cycles [21, 13, 19, 12, 4]. Other authors such as Suiker [22], Karg [16], Malek [1] have worked with more detailed elastic-plastic material models to simulate the permanent settlement and deterioration from cyclic loads.

In this study, we have carried out a study of the existing empirical settlement laws of ballast track, then selecting an appropriate vehicle-track model through a full analysis of the dynamic interaction of 2D and 3D models using FEM. The appropriate vehicle-track model will be used

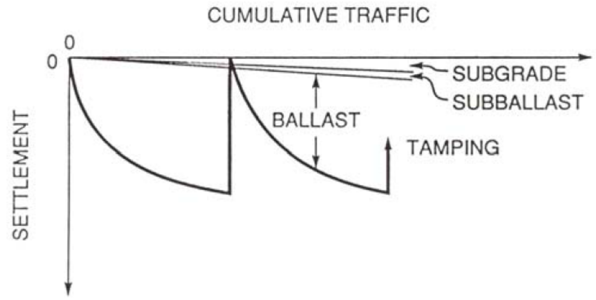


Figure 1: Substructure contributions to settlement (from [20]).

for a study of prediction of the track settlement incorporating a settlement law to evaluate the dynamic effect on the ballast track settlement. The results obtained including the track profile growth and the contact force will be discussed and some future works will be suggested.

2 TRACK SETTLEMENT LAWS

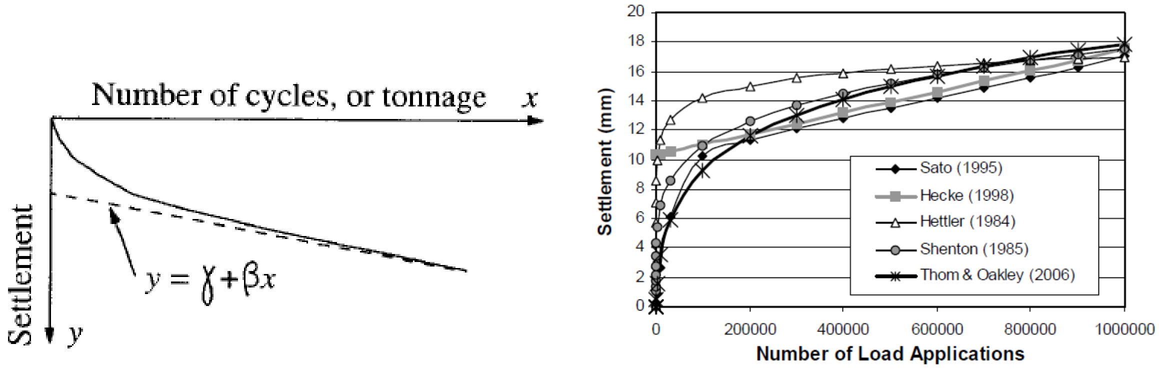
The railway track will settle as a result of permanent deformation in the substructure such as ballast, subballast and subgrade. The settlement of ballasted track occurs in two major phases:

- Phase I: is relatively fast, usually occurs in the process of consolidation of ballast. During this phase, the ballast is compressed to achieve high density.
- Phase II: after the first phase, the increment of settlement is slower, and with a more or less linear relationship between settlement and time (or number of loading cycles).

The second phase of settlement is caused by several basic mechanisms of substructure behavior:

1. There is a continuous volume reduction, i.e. densification caused by particle rearrangement produced by repeated train loading.
2. Subballast and/or subgrade penetration into ballast voids takes place.
3. There is volume reduction caused by particle breakdown from train loading or environmental factors, by abrasive wear.
4. Movement of ballast and subgrade particles away from under the sleepers causes the sleepers to sink into the ballast/subgrade.
5. Lateral and possibility also longitudinal, movement of sleepers causes the ballast beneath the sleepers to be "pushed away", and the sleepers will sink deeper into the ballast.

Dahlberg [10] has performed a critical summary covering existing settlement models to simulate the deterioration or settlement of the ballast track. The first model was introduced by Shenton [21] through the results of both laboratory and field experiments. A similar model was suggested by Sato [19]. Most authors have tried to introduce into their model the two phases of track settlement. The first phase is a nonlinear relationship between the settlement and the time



(a) Sato model [19]

(b) Comparison of settlement prediction models from the literature ([24])

Figure 2: Settlement prediction models.

(or number of loading cycles), while the second phase tends to be linear. As the objective of this work is to predict the track settlement in function of the dynamic forces, we have examined some track settlement models which depend on the dynamic force such as Guérin model [12] that will be explained in more detail in the following subsection.

2.1 Guérin law (1996)

In an extensive study by Guérin [12], a laboratory test reduced to a scale(1-3) called "Microballast" was used to investigate the ballast and subgrade settlement. Guérin's work is based on the model proposed by Shenton [21]. Track settlement evolution is divided into two phases. In the first phase, mainly ballast compaction occurs, giving a ballast settlement rate that is relatively large. In the second phase the model includes a steady state settlement rate. In this phase, the settlement rate (τ) per loading cycles (N) is expressed as a function of the maximum elastic deflection (d) of the ballast and subgrade during the loading cycle:

$$\frac{\mathrm{d}\tau}{\mathrm{d}N} = \alpha d^{\beta} \tag{1}$$

where α and β are material parameters. For the material used in the experiment reported, Guérin has determined the value of these parameters: α=0.48$\times$10^{-6} and β=2.51 (with a coefficient of correlation of 0.61, indicating some scatter in the data). This model was also validated by Bordin [4] through another experiment similar to "Microballast". The two tests of Guérin and Bordin are valid when the train speed is lower than 350 km/h. In 2008 ,in order to study this model at a higher speed, Al-Shaer [2] has improved the experimental process and has validated Guérin's model with some changes in the coefficients involved.

Table 1: Coefficients α and β for the Guérin, Bordin and Al-Shaer experiments.

			Ali Al-Shaer			
			$v < 350$		$v > 350$	
	Guérin	Bordin	Normal soil	Soft soil	Normal soil	Soft soil
α	1.44×10^{-6}	2.50×10^{-6}	9.67×10^{-6}	3.05×10^{-6}	1.10×10^{-6}	1.33×10^{-6}
β	2.51	1.17	1.46	2.41	1.5	1.56

3 DYNAMIC ANALYSIS OF TRAIN-TRACK INTERACTION

3.1 Vehicle Models

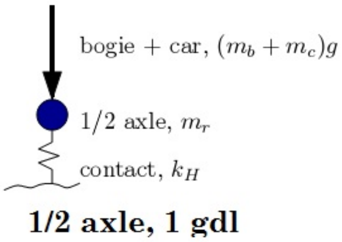
Railway vehicles are complex mechanical multibody systems, including rigid body, linear and nonlinear springs and different types of damping that can be analyzed and designed using modern computational mechanics techniques. To simulate the dynamic vehicle-track interaction, several vehicle models have been used in research: from simple models such as a moving load to more complex models of multibody system in 3D. Often it is undesirable to employ sophisticated and complex vehicle models which are not well understood and whose details play no role in the vertical dynamic loads transmitted to the track infrastructure which is the objective here. The type of models to employ must be well understood and adequately selected. In this study, we developed different vehicle models in 2D and 3D, which takes into account the mass that vibrates with the deformation of the track.

3.1.1 2D vehicle model

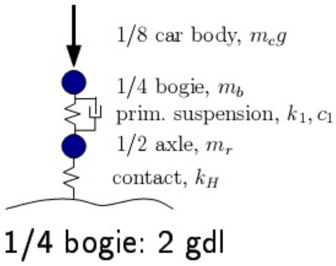
For the 2D analysis, we have considered 5 models of train: from the simplest half axle model (or moving mass) to a more complex model of half vehicle (see figure 3). The vibrating masses considered are the masses of the wheel, bogie and train body. Depending on the model, primary and secondary suspensions consisting of discrete springs and dampers are also taken into account. In these models, the contact between wheel and rail is considered as a Hertz's nonlinear spring. It was considered that the rail and wheel are the same material with the elastic modulus E and Poisson's ratio ν, using Hertz's normal elastic contact theory ([15]) the nonlinear relationship between the vertical contact force F_v and the vertical relative deformation δ_v is given by the relation (2).

$$F_v = \delta_v^{3/2} C_H \ \text{ where } C_H = \frac{2E}{3(1-\nu^2)}(r_r r_w)^{1/4} \tag{2}$$

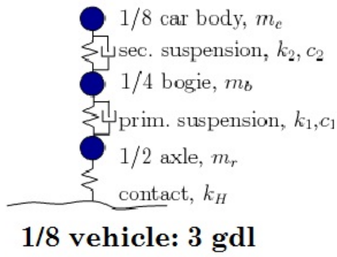
Figure 4 shows the displacement of a point in the rail obtained in the dynamic calculation of vehicle-track interaction with different models considered. It is noted that the structural responses obtained are very similar. However, the 1/2 axle model does not represent the reality of the vehicle, this model only has one natural frequency (about 220 Hz) while the actual vehicle has other relevant frequencies that could give different results in other frequency of excitation by the irregularities, speed, etc ... The model of 1/4 bogie gives results similar to other models



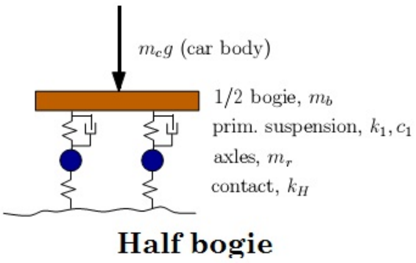
(a) 1/2 axle model



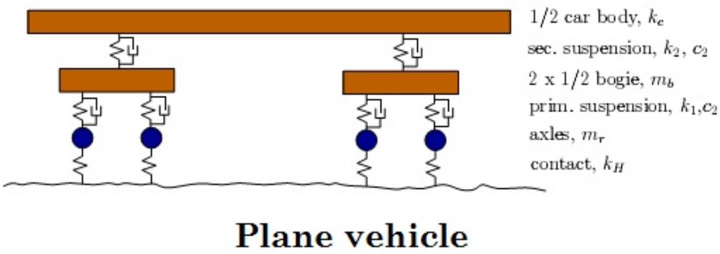
(b) 1/4 bogie model



(c) 1/8 vehicle model



(d) 1/2 bogie model



(e) 1/2 vehicle model

Figure 3: Two-dimensional vehicle models.

and has natural frequencies in the range of interest (f_1 = 220.9 Hz, f_2 = 3.82 Hz) compared to other excitation frequencies of the track. Therefore the 1/4 bogie model was used for this work.

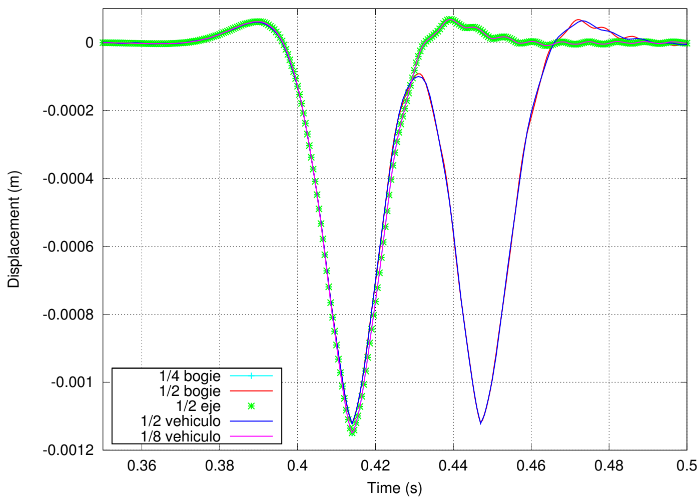


Figure 4: Displacement of a point of rail for different vehicle models at speed 360 km/h.

3.1.2 Full three-dimensional model of the vehicle

The vehicle is modeled using a multibody system with mechanical properties corresponding to the ICE 3 high speed vehicle (AVE S103) (figure 5). The considered vehicle model includes the box, bogies and wheelsets as rigid bodies with associated mass and inertia. Each rigid body has 6 degrees of freedom (DOF). The bodies are connected by two levels of suspension: primary and secondary. The suspension elements are modeled using springs and dampers with linear behavior. We have studied the modes of vibration of the vehicle modeled and the frequencies of the modes are obtained in the range of 0 to 40 Hz. Table 2 presents the first most representative modes of vibration of the vehicle.

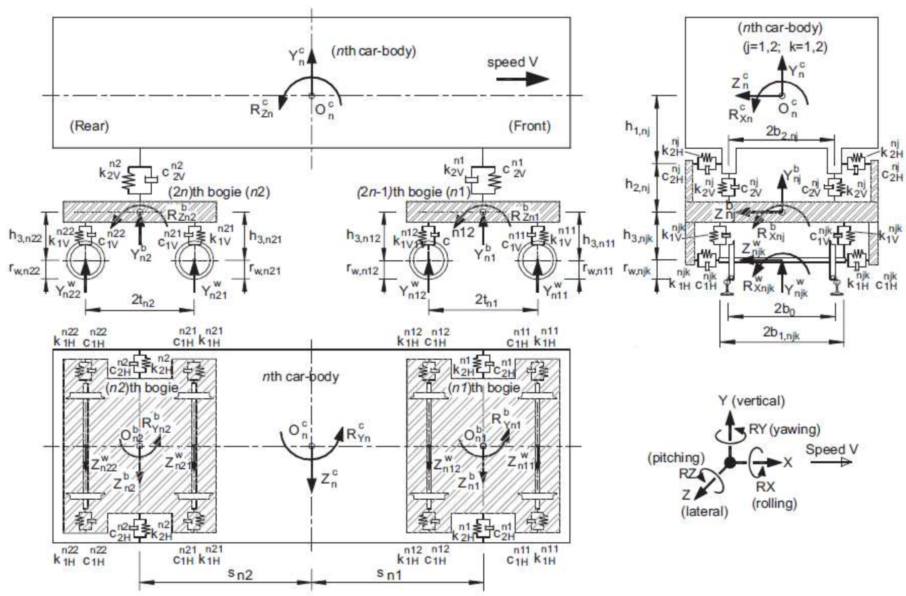
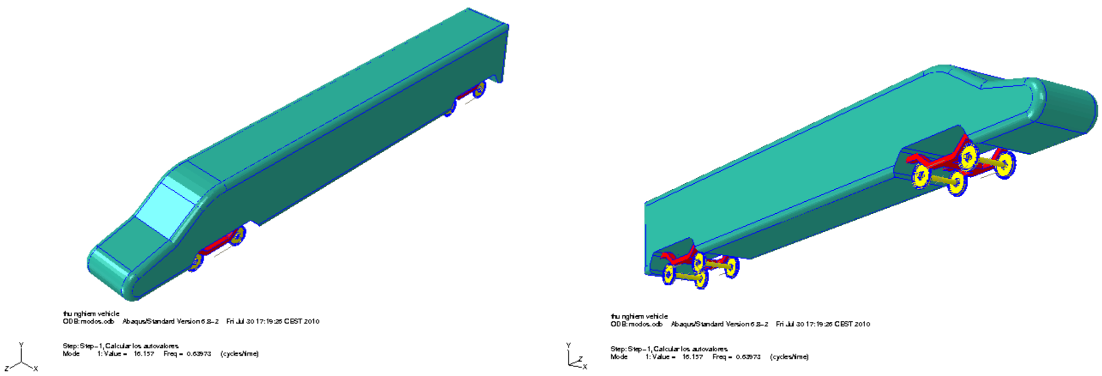


Figure 5: Full three-dimensional vehicle (ICE3) models.

Table 2: Frequencies of vibration of vehicle model developed in Abaqus.

Vibration modes		
No. of mode	Frequency (Hz)	Description
1	0.63973	Lateral movement and rolling car-body
2	0.75975	Vertical movement of car-body
3	0.94684	Pitching car-body
4	1.12670	Rolling bottom of car-body
5	3.28060	Yawing car-body and rolling bogies

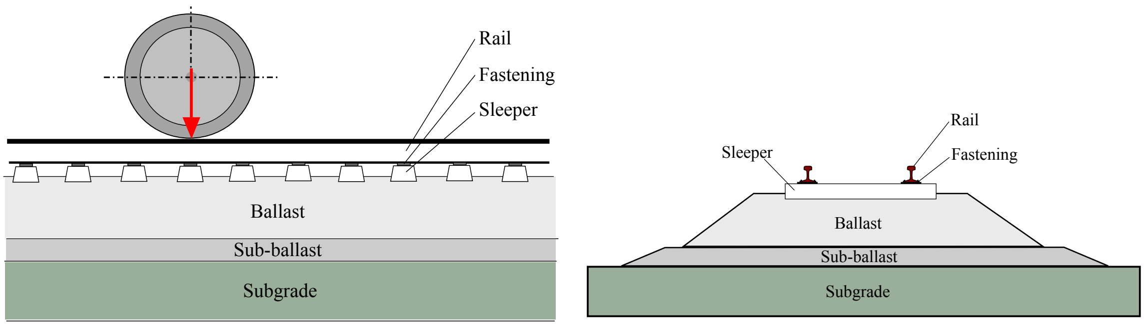


(a) (b)

Figure 6: Model developed in Abaqus.

3.2 Track models

Generally, the structure of ballast track is composed by rail, railpads, sleepers, the ballast layer, the possible sub-ballast layer, the subgrade (see figure 7). The ballast track models in 2D and 3D have been developed.



(a) Longitudinal section (b) Transversal section

Figure 7: Structure of ballast track.

3.2.1 2D model

In 2D, the track is modeled by beam, truss and 2D continuum finite elements (figure 8). These types of models have been used in other previous works [5, 23, 17, 14]. The rail has been simulated as a continuous Timoshenko beam including shear deformation, supported by pads which are springs and dampers. The sleepers are regarded as a concentrated mass, the ballast is considered as spring and damper. The sub-ballast is not considered in this work. The subsoil is modeled as an infinite linear elastic spring. The track length studied is 90.0 m (table 3)

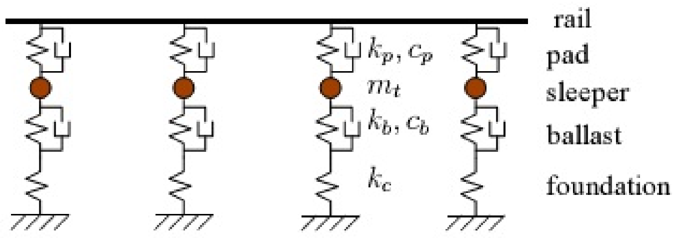


Figure 8: Ballast track model in 2D

Figure 9 presents the vertical dynamic response of track in the frequency domain. Figure 10

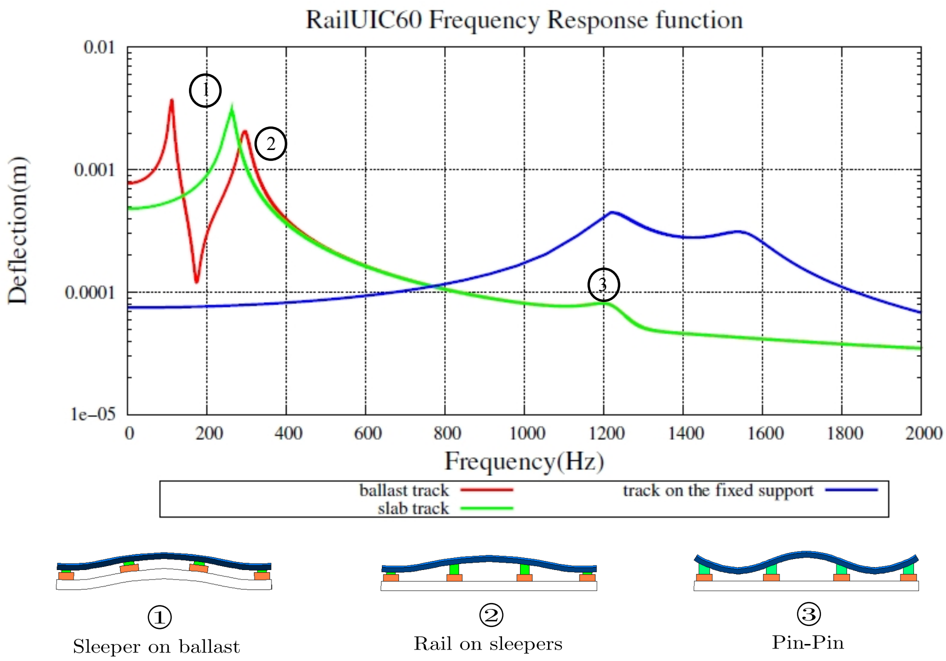


Figure 9: Vertical dynamic response of track.

presents the static response of track when the force is applied at the centre of the track length studied, between two sleepers.

3.2.2 3D Model

The track is composed by rail, railpads, sleepers, a ballast layer, a sub-ballast layer and a subgrade layer which are modeled by solid elements with linear elastic behavior. The mechan-

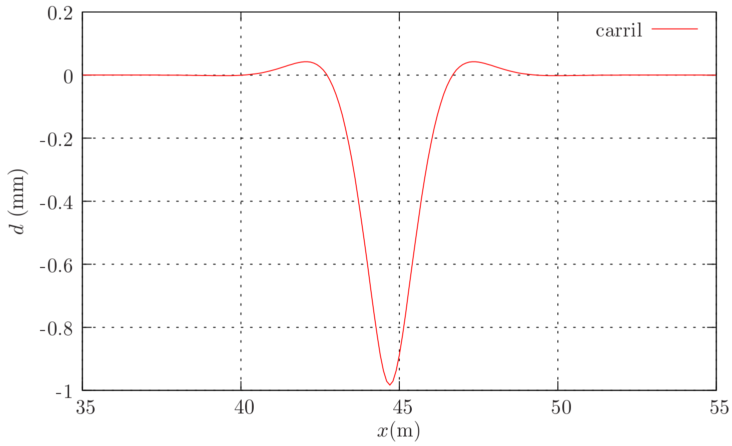


Figure 10: Static response of ballast track 2D.

Dimensions	
Model length	90.0 m
Ballast thickness	40 cm
Separation pads	0.60 m
Properties	
Rail	UIC60
Pads stiffness (k_p)	100 kN/mm
Damping pads (c_p)	0.015 kN s/mm
Ballast stiffness (k_b)	100 kN/mm
Damping ballast (c_b)	0.0123 kN s/mm
Mass of half sleeper ($m_t/2$)	160 kg
Point foundation stiffness (k_c)	80 kN/mm

Table 3: Model parameters of ballast track

ical properties of the materials are similar to the 2D model. To model the underlying soil is an important problem, as in principle a detailed 3D model should extend to infinity, in order to void reflection of shear and pressure waves transmitted from the structure. Of course practical considerations make this unfeasible. In this study we have applied infinite elements which provide simultaneously the impedance of the foundations and non-reflecting boundaries, as implemented in Abaqus based on the work of Zienkiewicz [25] (see figure 11).
Figure 12 shows the static response of track when two forces apply on the track with the distance between these two forces equal to the distance of two axles of bogie of ICE3.

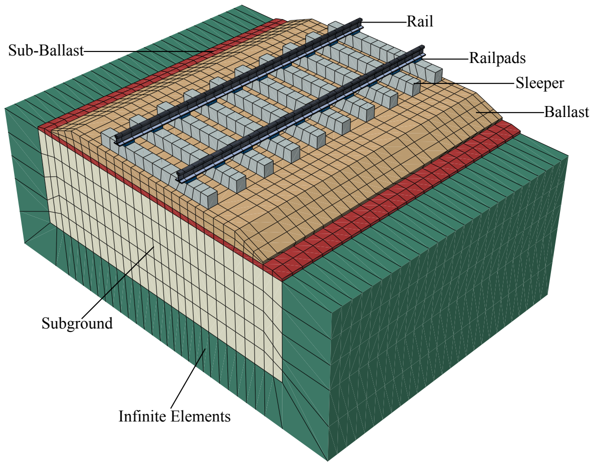


Figure 11: Ballast track model in 3D developed in Abaqus.

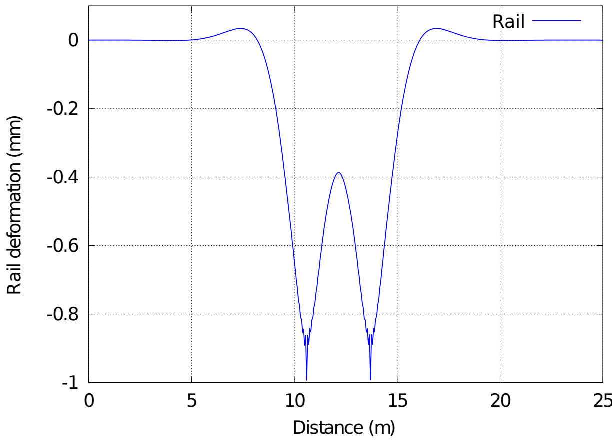


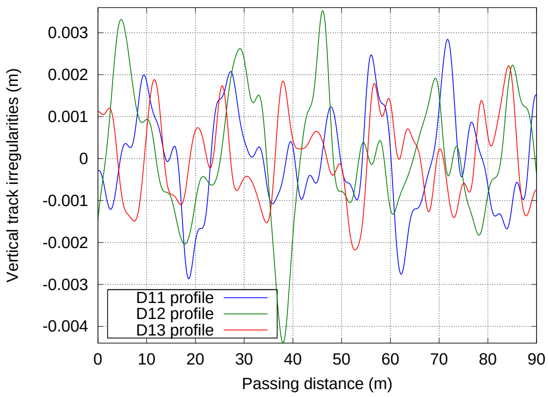
Figure 12: Vertical static response of ballast track 3D.

3.3 Vehicle-Track Interaction

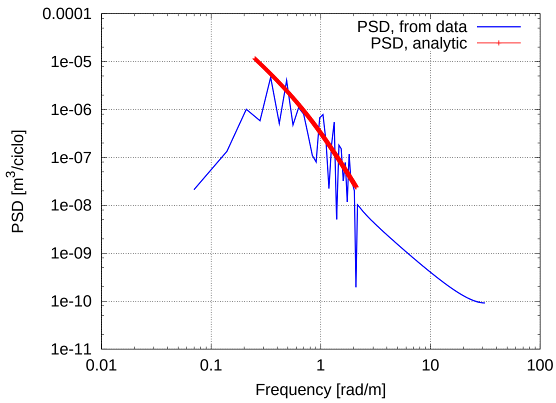
In this part of the study, the train runs on the track with constant speed, taking into account the track irregularity profile (the wavelength is in range [3m-25m]). The irregularity is generated from the power spectral density (see [9]) according to maximum considered limit (intervention limit) defined in [8]. For dynamic analysis, we have applied three different generated irregularity profiles consistent with such limits (figure 13(a)). And the reverse process has been applied to verify the accuracy of the irregularities created (see figure 13(b)).

3.3.1 2D dynamic interaction

As discussed in the previous section 3.2.1, the model of 1/4 bogie is selected for this work. Therefore, we used this model of 1/4 bogie to study the dynamic interaction of vehicle/track.



(a) Vertical irregularity profiles

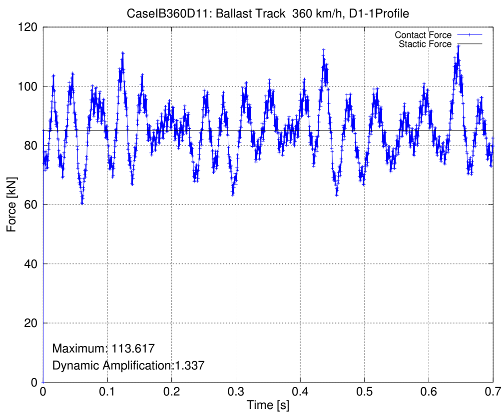


(b) Power spectral density

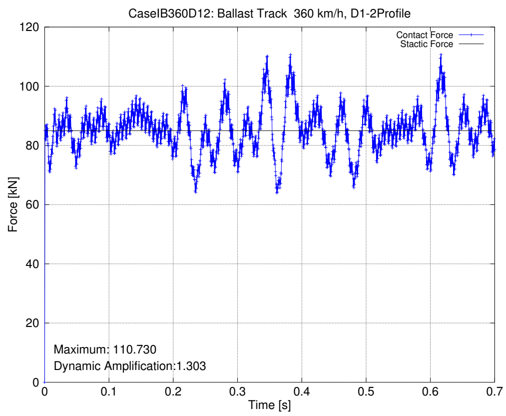
Figure 13: Generation of vertical irregularity profiles.

The calculation is done in the time domain, using the HHT time integration method to solve the transient problem. The contact problem is modeled by the method of Lagrange multipliers. The numerical simulations are done with different speeds (from 200 km/h to 360km/h) for each irregularity profile proposed and we have obtained the following results:

- Contact force between the wheel and the rail.
- Envelope of dynamic amplification of contact force in function of the speed.



(a) D11 profile



(b) D12 profile

Figure 14: Contact force in different profile at the speed 360 km/h.

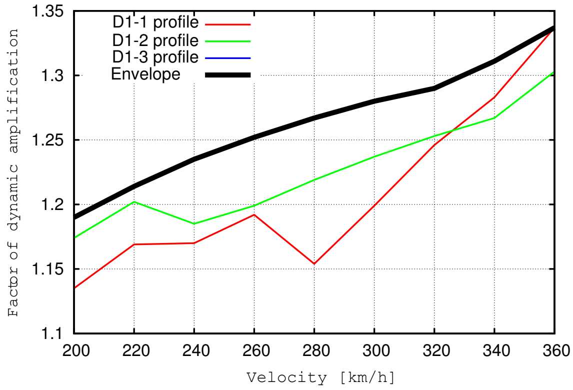


Figure 15: Envelope of dynamic amplification of contact force

3.3.2 3D dynamic interaction

The analyses have been carried out using the 3D coupled vehicle/track model (figure 16). We obtained the contact force and the envelope of dynamic amplification of contact force in function of speed. The results obtained will be compared with the results in 2D dynamic interaction 3.3.1. Some representative results are shown in figure 17. We may observe that the results

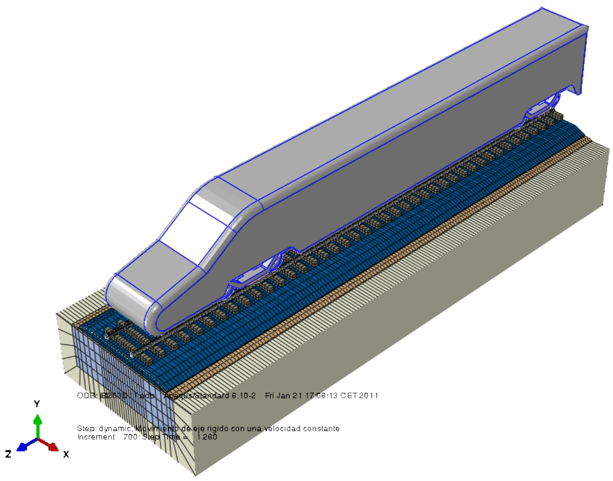
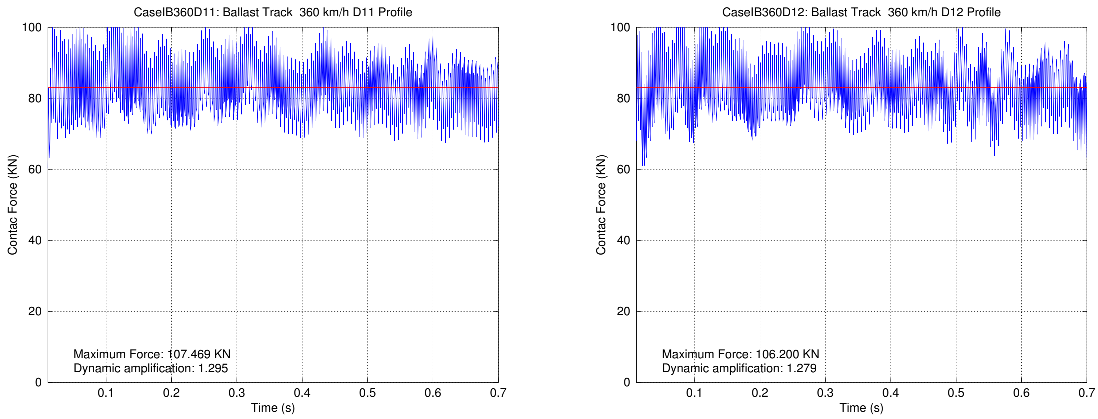


Figure 16: Three-dimensional coupled vehicle/track model.

obtained in the 3D dynamic analysis are similar in amplitude to 2D results. Demonstrating the validity of 2D model used. For the reason of the calculation time, the 2D model was calculated about two minutes, while the 3D model took about 35 hours. As a result we selected the 2D model as a practical means for evaluation of dynamic loads for predicting the track settlement.



(a) D11 profile

(b) D12 profile

Figure 17: Contact force between wheel/rail at speed v=360km/h in 3D analysis.

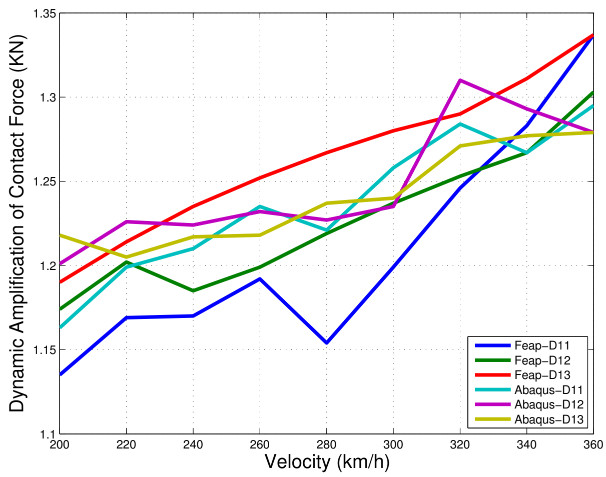


Figure 18: Comparison of the results obtained in 3D analysis with 2D analysis.

4 PREDICTION OF TRACK SETTLEMENT

In this last part of the work, a settlement model has been applied to predict the settlement of ballast track. To apply this model in the numerical simulation of vehicle/track in the time domain, we applied a calculation process which is represented in the figure 19. As has been mentioned, the vehicle/track model used for dynamic calculations has been a 2D track model with the vehicle model of 1/4 bogie. This model takes into account adequately the vertical dynamic loads which are main cause of track settlement. As initial condition, it is assumed that the track has some initial defects, a vertical profile of longitudinal irregularities. The analysis of the evolution of this track profile with the number of loading cycles for a given train speed is the objective of this work. The parameters used in the calculation process (figure 19) are:

- Increment of number of loading cycle ΔN = 50000 cycles.
- Maximum value of settlement: y_{max} = 5 mm.
- Maximum number of loading cycle: $N_{max} = 10^6$ cycles.

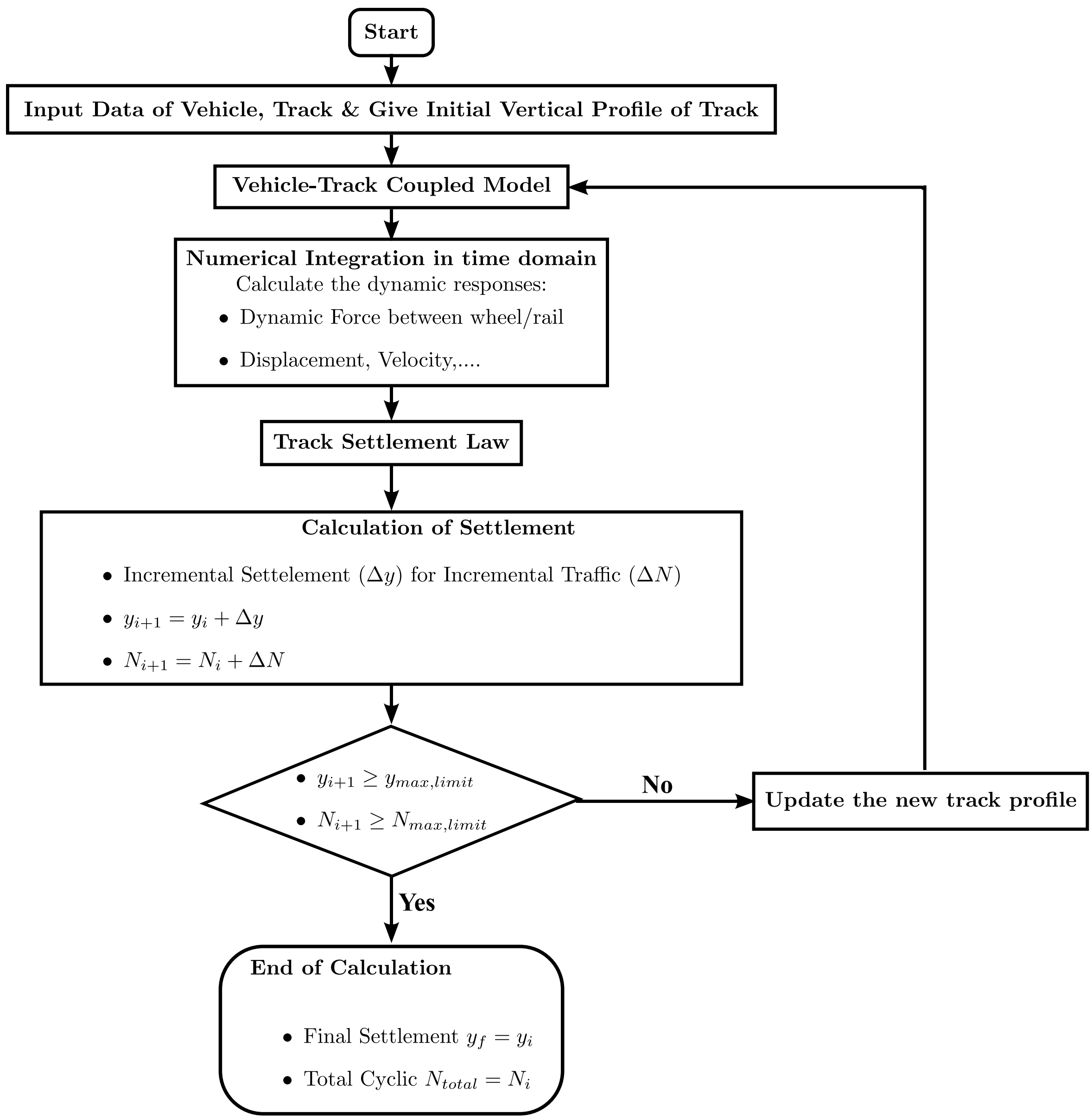
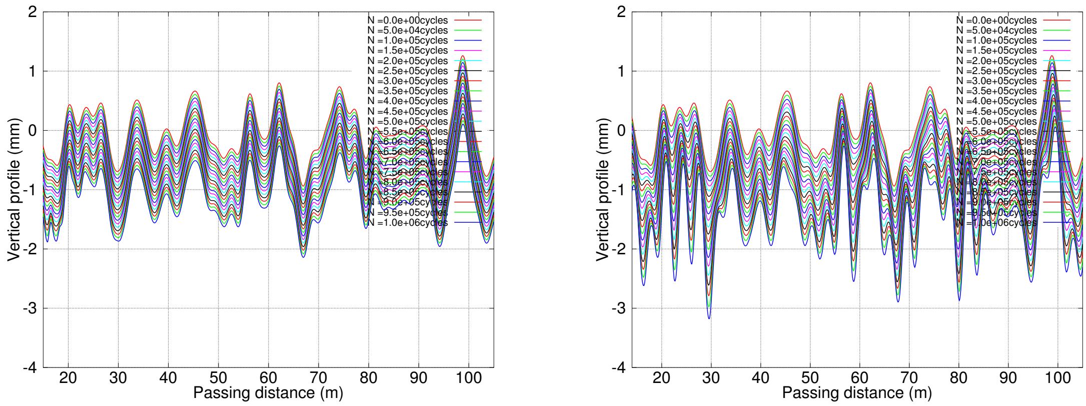


Figure 19: Flowchart of the calculation process applied in this work.

4.1 Prediction with Guérin model

Guérin's settlement model has been implemented in the vehicle/track model according to the calculation process (figure 19), in order to estimate the increment of the residual settlements in each loading cycle which is a function of the parameters of track response to passing trains. The maximum elastic deflection of the ballast, d in equation (1), is computed for the passage of a 1/4 bogie. Assuming ΔN such cycles have passed, the new profile of the rail and track is updated, adding the settlements predicted at each point, before a new time simulation takes place. This is done for every interaction i, with ΔN cycles, until calculations reach a given maximum settlement (in any point of track) or reach a given total number of loading cycles. The calculations were carried out in the dynamic model with different train running speeds (from 200 km/h to 360km/h).

Figure 20 presents the evolution of settlement with the increasing number of cycles for train speeds of 200 km/h and 360 km/h respectively. It may be noticed that for 360 km/h the settlement produced is greater. This result is consistent with what we obtained in the dynamic analysis in the previous section 3.3.1. In fact, when the train velocity increases the dynamic force increases too, producing correspondingly greater elastic deflection of ballast.



(a) Velocity of 200 km/h

(b) Velocity of 360 km/h

Figure 20: Track profile evolution with Guérin law

Figure 21 shows the settlement evolution in a point of track. It may be noticed that there is a non-linear relationship between the settlement and the defection d, and the settlement evolution depends on the dynamic effect produced.

Figure 22 shows the contact force for train speeds of 200 km/h and 360 km/h. It can be seen that the contact force increases substantially when the track defects and speed are more important.

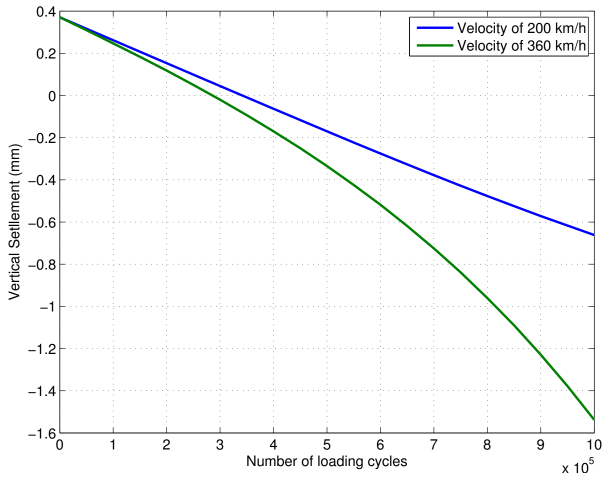
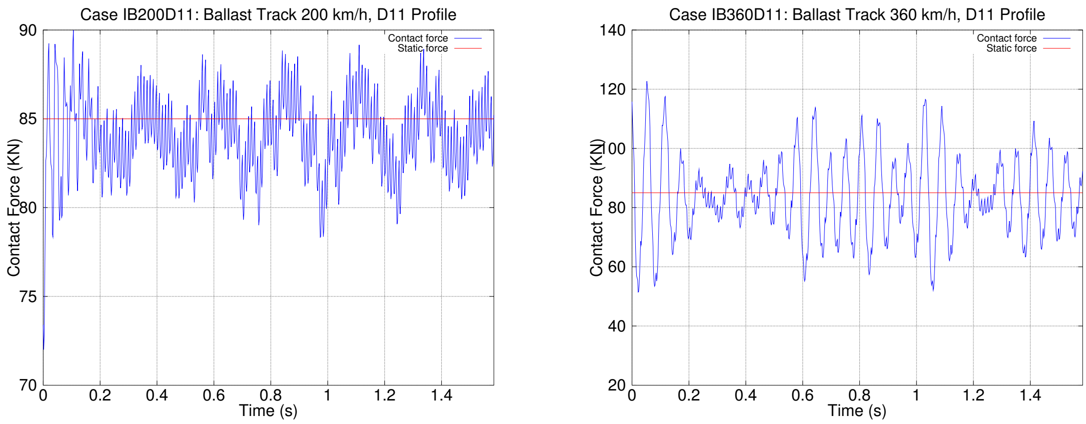


Figure 21: Settlement evolution with different velocity



(a) Velocity of 200 km/h

(b) Velocity of 360 km/h

Figure 22: Contact Force

5 CONCLUSIONS

In this study, several vehicle/track finite element models have been analyzed and Guérin law has been incorporated in the numerical simulation to predict the evolution of track settlement. The results obtained indicate the influence on track irregularity evolution of the dynamic vehicle/track interaction. It is shown how the long-term behavior of ballast track may be predicted in detail from the dynamic loading of the track (number of loading cycles) an on the ballast deflection due to that load. An increase of train speed will produce higher contact forces between the wheel and the rail, and will produce larger deflections in the ballast and a larger settlement will be obtained.

Simplified 2D models for track and vehicle may be employed for this purpose. Ballast is modeled as linear springs and dampers and only takes into account the vertical effect, neglecting the longitudinal and transversal effects. For the future work, the model can possibly be improved by extending it to 3D with material model based on hypoplasticity [3], or generalized plasticity [18], or Suiker's material [22] to describe the long-term behavior of ballast.

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